1.2: Complex Numbers

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CHAPTER 1: COMPLEX NUMBERS IN "The Mathematics of Quantum Mechanics"

1.0: Introduction

- Real numbers ($\mathbb{R}$)

1.1: What is a complex number?

- Imaginary unit number (i): square root of -1

A number is imaginary if its square is negative.

- Complex number: any # written in form $z = a + bi$, where a & b are real #s.

$z = a + bi$ ← imaginary part

↑ real part

$Re(z) = a$

$Im(z) = b$

- Complex plane: a visulation of complex numbers using the x-axis to denote the real part, and the y-axix to denote the imaginary part.

1.2: Doing math with complex numbers

- Complex addition: $z + w = (a + bi) + (c + di)$

$z + w = (a + c) + (b + d)i$

- Complex multiplication: $zw = (a + bi)(c + di)$

$zw = ac + adi + bci + bdi$

$zw = (ac - bd) + (ad + bc)i$

- Complex conjugate: the complex conjugate of complex number $z = a + bi$ is $\bar{z} = a - bi$.
- Modulus of a complex number $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$.
  - ↳ It is always real and positive.
  - ↳ Distance from origin to complex number on complex plane.
  - ↳ $a^2 + b^2 \Rightarrow |z|^2 \Rightarrow \sqrt{z\bar{z}}$
- Complex division: $\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{a}{a^2+b^2} - \frac{bi}{a^2+b^2}$

SUMMARY OF PROPERTIES

$z + w = w + z$

$zw = wz$

$\overline{z + w} = \bar{z} + \bar{w}$

$\overline{zw} = \bar{z}\bar{w}$

$z\bar{z} = \bar{z}z = |z|^2$

$\bar{\bar{z}} = z$

$|z| = |\bar{z}|$

$|zw| = |z||w|$

$|z + w| \leq |z| + |w|$

$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{|z|^2} \quad z \neq 0 + 0i$

1.3: Euler's formula and the polar form

- Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$, where θ is a real number in radians.
  - ↳ any complex number $z = a + bi$ can be written in the form $z = |z|e^{i\theta}$, where θ is the angle between the real axis and complex # in the complex plane.

    ↖ polar form
  - ↳ $\theta = \arctan\left(\frac{b}{a}\right)$, $\theta = \arcsin\left(\frac{b}{|z|}\right)$, & $\theta = \arccos\left(\frac{a}{|z|}\right)$
  - ↳ θ is known as the argument of the complex number
- Periodicity: the function $e^{i\theta}$ is a periodic function of θ with a period of 2π: $e^{i(\theta \pm 2\pi)} = e^{i\theta}$
  - ↳ θ always has a value between 0 and 2π
- The complex conjugate of $|z|e^{i\theta}$ is $|z|e^{-i\theta}$
- Summary of properties of the polar form:

$e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)} \rightarrow zw = (|z|e^{i\theta})(|w|e^{i\phi}) = |z||w|e^{i(\theta+\phi)}$

$(e^{i\theta})^n = e^{in\theta}$ (for any number n)

$\frac{1}{e^{i\theta}} = (e^{i\theta})^{-1} = e^{-i\theta}$

$|e^{i\theta}| = e^{i\theta} \cdot \overline{e^{i\theta}} = e^{i\theta}e^{-i\theta} = e^{i(\theta-\theta)} = e^0 = 1$

$\overline{e^{i\theta}} = e^{-i\theta}$

since $e^{\pm 2\pi i} = \cos(\pm 2\pi) + i\sin(\pm 2\pi) = 1$, then $e^{i(\theta \pm 2\pi)} = e^{i\theta} \cdot e^{\pm 2\pi i} = e^{i\theta}$

PAPER: "Why are complex numbers needed in quantum mechanics?"

- Quantum mechanics deals with complex quantities of a special kind, which cannot be split into real & imaginary parts that can be treated seperately
  - ↳ wave function & quantum state vectors
  - ↳ matrix: $([\hat{p}, \hat{x}] = -i\hbar)$
  - ↳ wave formulation $(i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi)$
- Justifications for the need of complex numbers
  1. The impossibility of having information on position when momentum is exactly known
  2. The fact that i appears explicitly in the Schrödinger equation
  3. The descriptions of S_x, S_y, and S_z in sequential Stern-Gerlach experiments
  4. The demand for continuous transitions

SECTION 1: COMPLEX NUMBERS IN QUANTUM MECHANICS IN "Illinois Course Phy 580"

- i is everywhere in quantum mechanics
  - ↳ Heisenberg's commution relation: $QP - PQ = i\hbar$
  - ↳ Schrödinger's equation: $\frac{\hbar}{i}\partial_t \Psi = H\Psi$