1.3: Linear Algebra Basics

Vectors

Most of Quantum Mathematics is linear algebra based, so it is absolutely essential to grasp the basics before moving on to more complex and niche mathematical topics. To quote The Mathematics of Quantum Mechanics, “the language of quantum mechanics - linear algebra”.

Vectors are at the center of linear algebra. Vectors, basically, are a way of representing something that can’t be described with a single number. Think about a coordinate system for example: A vector can be shown visually on the graph as an arrow starting at one point and going to another. For simplicity’s sake, let’s always place the ”tail” (non-arrow) side of the vector on the origin. The point that the arrow lands on, let’s call (x_1, y_1). To refer to this point, we have to use 2 numbers. Or, we can use the vector notation: $\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \right)$. So, in essence, a vector is a way of representing multiple numbers.

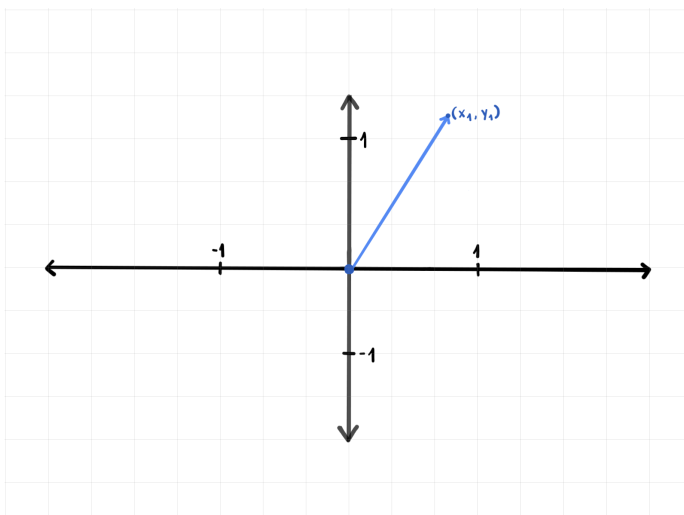

Figure 1: Vector $\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \right)$.

There are two types of vectors: row vector and column vectors. The vector above is a column vector, which is a single column of numbers: $\left(\begin{bmatrix} x_1, y_1 \end{bmatrix} \right)$. A row vector, like a column vector, is exactly what it sounds like - a row of numbers: $\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \right)$. Each number in a vector $\overrightarrow{v}$ is a component of $\overrightarrow{v}$, with the first number either in the row or column being the first component, v_1, the second number being the second component, v_2, and so on and so forth until the ith component, v_i. The total number of numbers, or components, in a vector is referred to

as the dimension of the vector. So, for instance, vector $\overrightarrow{v} = (\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix})$ has degree 3.

Vectors, just like the numbers we use everyday, can be added, multiplied and otherwise manipulated. The rules, however, differ in some cases.

Vector addition is the process where two or more vectors are added together. If $\overrightarrow{v} = (\begin{bmatrix} v_1 \\ v_2 \end{bmatrix})$ and $\overrightarrow{w} = (\begin{bmatrix} w_1 \\ w_2 \end{bmatrix})$, then $\overrightarrow{v} + \overrightarrow{w} = (\begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix})$. This can be abstracted to if $\overrightarrow{v}$ are arbitrary n-dimensional vectors, the jth component of $\overrightarrow{v} + \overrightarrow{w}$ denoted $(\overrightarrow{v} + \overrightarrow{w}_j$ is $v_j + w_j$.

Scalar multiplication happens between a scalar (such as 2 or π) and a vector. If $\overrightarrow{v} = \binom{v_1}{v_2}$ and c is a scalar, $c\overrightarrow{v} = (\begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix})$. From a component view, for vectors of any dimension, $(c\overrightarrow{v})_j = cv_j$. Scalar multiplication of a vector by a positive integer will not change the direction of the vector, but scalar multiplication by a negative integer will invert a vector's direction.

The magnitude, or norm, of a vector ($|\vec{v}|$) is the length of that vector. Magnitude can be calculated using Pythagorean Theorem, as a vector can be thought of as the hypotenuse of a triangle where the x distance and y distance are the other two legs. The Pythagorean Theorem works for any number of dimensions, so it will work no matter how many elements the vector in question has. Therefore, $|\vec{v}| = \sqrt{v_1^2 + v_2^2 ... v_n^2}$. A vector with length 1 is called a unit vector. Any nonzero vector can be "normalized", or scaled to a unit vector. To normalize a vector $\vec{v}$, multiply it by $\frac{1}{|\vec{v}|}$.

Linear independence is a quality of a group of vectors of the same dimension. If a set of vectors is linearly independent, no vector from the set is a scalar multiple of another. Mathematically, if c and d are nonzero scalars and $\vec{v}$ and $\vec{w}$ are vectors of the same dimension, then $\vec{v}$ and $\vec{w}$ are linearly independent if and only if $c\vec{v} \neq d\vec{w}$.

Lastly for vectors is **vector space**, which is the collection of all the complex numbers that can be formed from a set of vectors using vector addition and scalar multiplication. In order for a set (a collection of mathematical objects) to be considered a vector space, the set must be closed under addition scalar multiplication, meaning any vector generated via scalar multiplication and vector addition using vectors in the set with also be in the set.

Matrices

Simply, a matrix is just a box of numbers. Matrices are very similar to vectors, and share a lot of the same properties; however, they can have both multiple rows and multiple columns: ($\begin{bmatrix} a, b \\ c, d \end{bmatrix}$). To indicate an element of a matrix, that element's row and column in the matrix is used. So, for any matrix Q, Q_{ij} is the number in the ith row and jth column. If a matrix has m rows and n columns, it is a mxn dimensional matrix.

Matrices can be added and multiplied by scalars the exact same way vectors are. However, the process to multiple two matrices becomes a little more difficult. There are several requirements for matrix multiplication. For the case of MxN, M must have the same number of columns as N has rows. Additionally, the multiplication between the matrices is not communicative, and therefore MxN is much different than NxM. The multiplication itself follows the rule $(MN)_{ij} = \sum_{k=1}^{n} M_{ik}N_{kj}$. To the mechanics of this, see Figure 2 below.

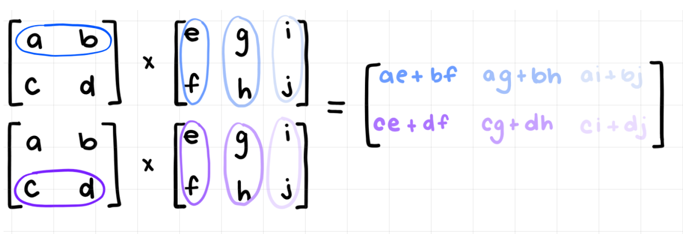

Figure 2: Matrix Multiplication

There are several special matrices that we will use later on: the identity matrix and the unitary matrix. The identity matrix where every element is zeros except for the left top to bottom right diagonal. The more formal definition is a matrix $\mathbf{I}$ defined such that for every nxn matrix M and any $\vec{v}$ in $\mathbf{C}^n$, $\mathbf{I}M = M\mathbf{I} = M$ and $\mathbf{I}v = \vec{v}$. Some examples include: ($\begin{bmatrix} 1, 0 \\ 0, 1 \end{bmatrix}$), ($\begin{bmatrix} 1, 0, 0 \\ 0, 1, 0 \\ 0, 0, 1 \end{bmatrix}$), and ($\begin{bmatrix} 1, 0, 0, 0 \\ 0, 1, 0, 0 \\ 0, 0, 1, 0 \\ 0, 0, 0, 1 \end{bmatrix}$). Identities are special because, as pointed out by the formal definition, because, when multiplied by a matrix with the same number of rows, it does not change the matrix. Same goes for a vector - it will not alter the vector.

The second special matrix is the unitary matrix, U, which is a matrix that satisfies $UU^{\dagger} = U^{\dagger}U = \mathbf{I}$, where signifies a complex conjugate.

Matrices end up being very important in Quantum Mechanics (and Linear Algebra overall) due to the fact that they can be thought of as linear functions! A linear function, f is a function that satisfies: 1. $f(x+y) = f(x) + f(y)$ for any input x and y and 2. $f(cx) = cf(x)$ for any input x and any scalar c. Matrices satisfy both of these requirements, and therefore, serve as linear functions when manipulated.

Dot Product and Basis

The dot product, or inner product, is way of combining two vectors. The dot product works by summing up the product of the corresponding elements in each vector: $\vec{v} \cdot \vec{w} = \sum_{j=1}^{n} v_j w_j$. The two vectors (in this case $\vec{v}$ and $\vec{w}$) must have the same dimension. Another way of computing the dot product of two vectors is with the magnitudes of each vector: $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta$, where θ is the angle between the two vectors. Because the dot product involves the angle, computing it can reveal information about the two vectors. Most notably, if the dot product of two vectors is zero, then the two vectors must be perpendicular, as $\cos 90 = 0$.
The dot product can be used for projection of one vector onto another. Projection is visually explained in Figure 3 below. The projection of vector $\vec{v}$ onto vector $\vec{w}$ is the vector along $\vec{w}$ until point on $\vec{w}$ that corresponds with a line perpendicular to the x-axis dropped down from the end of $\vec{v}$.

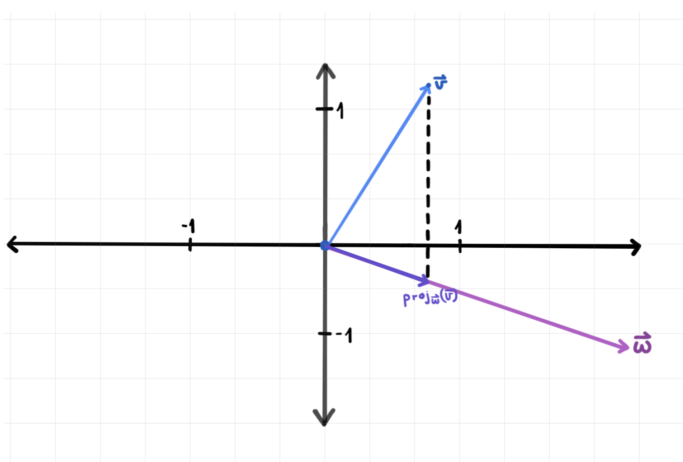

Figure 3: Vector Projections

Projection can be calculated using dot product and norms: $P_{\vec{v}\vec{w}} = \frac{1}{|\vec{w}|}\vec{w} \cdot \vec{v}$. $P_{\vec{v}\vec{w}}$ is a scalar and gives the component of the projection along $\vec{w}$ - the magnitude.

A basis is a finite set of vectors that can be used to describe any other vectors of the same dimension. Essentially, a basis is a group of vectors that can form any other vector with the same number elements through linear combination (vector addition or scalar multiplication). In order to qualify as a basis, the group of vectors must be linearly independent. The most familiar basis in 2 dimensions is the standard basis: $\vec{v} = (\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}) = v_1 (\begin{bmatrix} 1 \\ 0 \end{bmatrix}) + v_2 (\begin{bmatrix} 0 \\ 1 \end{bmatrix})$. An orthogonal basis is a basis where each vector has a magnitude of 1 and every vector is perpendicular (or orthogonal) to every other vector. The basis example above is a orthogonal basis because $(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$ and $v_2 (\begin{bmatrix} 0 \\ 1 \end{bmatrix})$ are perpendicular, as their dot product is equal to 0.

Vectors and matrices are at the heart of linear algebra, as well as form the core of quantum mechanics. Now that we've established a strong background in several of the linear algebra concepts, we can move on to the basics of quantum mechanics!