1.3: Linear Algebra Basics

Monday, September 5, 2022 3:27 PM

CHAPTER 2 : LINEAR ALGEBRA IN "The Mathematics of Quantum Mechanics"

2.0 : Introduction

- "the language of quantum mechanics - linear algebra"

2.1 : Vectors

- Vectors : just a way of stacking #s
  - ↳ Eg. $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ or $\begin{bmatrix} \frac{1}{3} \\ -3 \end{bmatrix}$ (both of these are column vectors)
  - ↳ Eg. $[5 \quad 2.1]$ or $[-3 \quad \frac{1}{2} \quad 4]$ (both of these are row vectors)
  - ↳ def: a column of numbers
  - ↳ the # of #s is known as the dimension of the vector
  - ↳ v_1 is the first component of $\vec{v}$, v_2 the 2nd, and so on
- A n-dimensional real vector lies in $\mathbb{R}^n$. A complex n-dimensional vector lies in $\mathbb{C}^n$.
- Vector addition : if $v_1 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\vec{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$, $\vec{v} + \vec{\omega} = \begin{bmatrix} v_1 + \omega_1 \\ v_2 + \omega_2 \end{bmatrix}$.
  - ↳ More generally : if $\vec{v}$ and $\vec{\omega}$ are arbitrary n-dimensional vectors, the j^{th} componant of $\vec{v} + \vec{\omega}$, denoted $(\vec{v} + \vec{\omega})_j$ is $v_j + \omega_j$
- Vector scalar multiplication : if $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and c is a scalar, $c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}$.
  - ↳ For vectors of any dimension, $(c\vec{v})_j = cv_j$
  - ↳ Scalar multiplication by a positive integer doesn't change it's orientation.
  - ↳ Scalar multiplication by a negative integer inverts it's direction.
- Vector space : the collection of all the complex #s of a given dimension with vector addition & scalar multiplication.
  - ↳ Abstract def: take a collection of mathematical objects (a set) with a well-defined addition & scalar multiplication. If
    1. the set is closed under addition & scalar multiplication, that's the result of adding 2 abitrary objects from the set, or the scalar multiplication of any objects, is also in the set;
    2. the set, the addition and scalar multiplication follow all properties.

2.2 : Matrices

- Matrix : a box of numbers.
  - ↳ Eg. $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
  - ↳ Given any matrix Q, Q_{ij} is the # in the i^{th} row & j^{th} column.
  - ↳ If a matrix has m rows and n columns, it is a $m \times n$ dimensional matrix.
- Matrix addition : $(M+N)_{ij} = M_{ij} + N_{ij}$
- Matrix scalar multiplication : $(cM)_{ij} = c(M_{ij})$
- Matrix multiplication : $(MN)_{ij} = \sum_{k=1}^{n} M_{ik} N_{kj}$
  - ↳ M must have the same # of columns as N has rows
  - ↳ Because of this property, matrices can be thought of as functions on vectors
  - ↳ Not commutative
- Linearity : a linear function (or a linear map, or a linear operator) f, is a function that satisfies :
  1. $f(x+y) = f(x) + f(y)$ or any input x and y.
  2. $f(cx) = cf(x)$ for any input x and any scalar c.
  - ↳ Matrices can be thought of as linear functions!

2.3 : Complex Conjugate, Transpose, and Conjugate Transpose

- Matrix/vector complex conjugate : if $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$, $M = \begin{bmatrix} c & d \\ f & g \end{bmatrix}$, then $\overline{\vec{v}} = \begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix}$, $\bar{M} = \begin{bmatrix} \bar{c} & \bar{d} \\ \bar{f} & \bar{g} \end{bmatrix}$.
  - ↳ General def : $(\bar{\vec{v}})_i = \bar{v}_i$, $(\bar{M})_{ij} = \overline{M_{ij}}$.
- Matrix/vector transpose : the transpose of matrix M, denoted M^t is such that the n^{th} row of M^t is the same as the n^{th} column of M.
  - ↳ Def : if $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$, $M = \begin{bmatrix} c & d \\ f & g \end{bmatrix}$, then $\vec{v}^t = [a, b]$, $M^t = \begin{bmatrix} c & f \\ d & g \end{bmatrix}$
  - ↳ General def : $(\vec{v}^t)_i = v_i$ $(M^t)_{ij} = M_{ji}$
  - ↳ The tranpose of a $m \times n$ matrix is a $n \times m$ matrix.
- Matrix/vector conjugate transpose : the transpose of the complex conjugate of a matrix/vector.
  - ↳ General def : $(M^\dagger)_{ij} = \bar{M}_{ji}$

2.4: Inner Products & Norms

- Inner Product: $\vec{v}\cdot\vec{w} = \sum_{j=1}^{n} \overline{v_j} w_j$
  - ↳ also called dot product or scalar product
  - ↳ The vectors must have the same dimension
- Hilbert Space: a vector space with a well-defined inner product (the collection of all n-d vectors with the inner product)
- Orthagonal Vectors: aka perpendicular - if the inner product of the 2 vectors is 0.
- Vector norm: aka the length of a vector - $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{\vec{v}^{\dagger}\vec{v}}$
  - ↳ A vector with norm 1 is called a unit vector.
  - ↳ $\|c\vec{v}\| = |c|\,\|\vec{v}\|$
  - ↳ Normalizing: scaling a nonzero vector $\vec{v}$ by $\frac{1}{\|\vec{v}\|}$ to make it have a unit length.

2.5 Basis

- Basis: a finite set of vectors that can be used to describe any other vectors of the same dimension. A set of n linearly independant vectors in $\mathbb{C}^n/\mathbb{R}^n$ is called a basis of $\mathbb{C}^n/\mathbb{R}^n$
  - ↳ ex. $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \underbrace{v_1}_{\text{coefficents}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \underbrace{v_2}_{\text{coefficents}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ form basis
- Linear Combination: a combination of any # of vectors using vector addition & scalar multiplication.
  - ↳ A set of vectors is linearly dependant if at least 1 of the vectors can be written as a linear combination of the others.
  - ↳ If a set of vectors isn't linearly dependant, they're linearly independant.
- Orthagonal Basis: a basis where each vector has norm 1 and each pair of vectors is orthagonal.
- Standard (canonical) basis: the basis used when explicitly writing a vector.
  - ↳ Also known as the computational basis

2.6: Inner Product as Projection

- Orthagonal projection: Given n-dimensional vectors $\vec{v}, \vec{w}$ in $\mathbb{C}^n$, the projection of $\vec{v}$ onto $\vec{w}$, $P_{\vec{v}\vec{w}} = \frac{1}{\|\vec{w}\|}\vec{w}\cdot\vec{v}$
  - ↳ The projection is given by inner product between the unit vector along $\vec{w}$ and $\vec{v}$
  - ↳ $P_{\vec{v},\vec{w}}$ is a scalar number (it's the component along $\vec{w}$)

2.7: Special Matrices

- Identity matrix: defined such that for every $n \times n$ matrix M, and any vector $\vec{v}$ in $\mathbb{C}^n$, $\mathbb{I}M = M\mathbb{I} = M$ and $\mathbb{I}\vec{v} = \vec{v}$.
  - ↳ denoted $\mathbb{I}$ (or sometimes I)
  - ↳ Preforms no action when operating - the output is always the same as the output.
- Unitary matrix: (U) a matrix that satisfy $UU^{\dagger} = U^{\dagger}U = \mathbb{I}$

CHAPTER 4: LINEAR ALGEBRA REVIEW... IN "Quantum Theory, Groups and Representations"

4.1: Vector Spaces & Linear Maps

- A basis (set of n linearly independant vectors) $\{e_j\}$, an arbitary vector $v \in V$ (a vector space) can be written as:
  $v = v_1 e_1 + v_2 e_2 + \dots + v_n e_n$ or $v = \begin{pmatrix} v_1 \\ v_2 \\ v_n \end{pmatrix}$
- Having basis $\{e_j\}$ allows the action of a linear operator L on V ($L: v \in V \to Lv \in V$) as matrix multiplication:

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \to \begin{pmatrix} L_{11} & L_{12} & \dots & L_{1n} \\ L_{21} & L_{22} & \dots & L_{2n} \\ \vdots & \vdots & & \vdots \\ L_{n1} & L_{n2} & \dots & L_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

4.2: Dual Vector Spaces

- Dual vector space: For V a vector space over a field k, the dual vector space V^* is the vector space of all linear maps $V \to k$, i.e.
  $V^* = \{\ell : V \to K \text{ such that } \ell(\alpha v + \beta w) = \alpha\ell(v) + \beta\ell(w)\}$ for $\alpha, \beta \in K$, $v, w \in V$.
- Transpose transformation: the transpose of L is the linear transformation $L^t : V^* \to V^*$ given by $(L^t \ell)(v) = \ell(Lv)$ for $\ell \in V^*$, $v \in V$.